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# ARTICLES

## Prospects for detecting the Christodoulou memory of gravitational waves from a coalescing compact binary and using it to measure neutron-star radii

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A coalescing compact binary, during its last tenth of a second of life, emits a burst of gravitational waves consisting of a high-frequency "chirp," with frequencies much greater than 100 Hz, superimposed on a gradually growing memory, known as the Christodoulou memory. Most of the memory's growth occurs over the last few hundredths of a second, so its signal has strong Fourier components at $f \sim 100$ Hz. The planned LIGO and/or VIRGO broadband gravitational-wave detectors have optimal performance at frequencies around 100 Hz and should be well suited, in terms of frequencies, to detect the growth of the memory amidst the chirp. If one or both of the binary's components is a neutron star (the other being either a neutron star or a black hole), then the growth of the memory will be cut off by the star's tidal disruption. The larger the neutron star's radius the sooner the cutoff and correspondingly the weaker the total memory. Therefore, from a LIGO and/or VIRGO measurement of the memory's strength, one could hope to infer the neutron-star radius. The prospects for such measurements to succeed are evaluated quantitatively and found to be poor because of the weakness of the memory. Even under optimistic circumstances the memory is so weak that only for a black-hole–black-hole binary is there much chance of detecting it, and then the prospects are only marginal.

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### I. INTRODUCTION

The continuing efforts to improve the sensitivity of gravitational-wave detectors, and the commencement of ambitious programs to build the kilometer-scale Laser Interferometric Gravitational Wave Observatory (LIGO) [1] and VIRGO [2] network of detectors, have encouraged attempts to predict the behavior of potential sources and types of gravitational waves. One important type is a gravitational wave with memory. These are waves which leave a system of free masses with a permanent relative displacement following their passage. Braginsky and Thorne [3] have discussed the optimal experimental strategy for detecting memories, and have estimated the sensitivities of a variety of detectors to waves with memory.

Until recently it was thought that waves with memory are restricted to sources whose constituent components are not gravitationally bound to each other, either initially or finally, or both. However, in 1991 Christodoulou [4] showed that strongly radiating bound systems, such as coalescing compact binary stars, have a significant memory created by nonlinearities in the wave-emission mechanism. This Christodoulou memory is actually the transverse-traceless part of the gravitational field produced by the stress-energy tensor of the gravitational wave itself [4–6]. Because, for a coalescing binary, it can be as large as one-tenth the size of the primary wave, the Christodoulou memory may be detectable by the planned network of gravitational-wave detectors, including LIGO [1] and VIRGO [2].

Coalescing compact binaries (whether NS-NS, NS-BH or BH-BH, where NS and BH mean neutron star and black hole) are expected to be among the strongest sources of gravitational waves for LIGO and/or VIRGO. Amongst the most interesting information that the experimenters might hope to extract from these binaries' waves is the neutron star mass-radius relation, since from it one can deduce the equation of state of matter at densities from nuclear to about ten times nuclear [7], which is little understood at present. Unfortunately, although waves from such a binary, in the frequency band of good expected LIGO and/or VIRGO performance (roughly 10–300 Hz), depend strongly on the binary's masses and might therefore allow fairly accurate mass measurements [8–10], they are insensitive to the radii of the binary's constituents. Strong dependence on the radii occurs only near the end of the inspiral, when the two objects are interacting tidally, merging and/or disrupting, and the frequencies of the primary waves are around a kilohertz. At these high frequencies the LIGO and/or VIRGO detectors will have relatively poor performance because of serious photon shot noise. In view of this, two different methods have been suggested for determining the radii [8]. One involves measurements of the kilohertz primary waves using specially configured narrow-band detectors. The other uses measurements of the primary waves' Christodoulou memory, detected by the same LIGO and/or VIRGO broadband detectors which will attempt the mass measurements. This paper and a companion one will evaluate these two methods, the memory method here and the narrow-band detector method in the companion paper [11]. As we shall see, the memory method is not very promising. In contrast, the narrow-band approach shows considerable promise.

The method discussed here relies on the following properties of the memory. The memory grows most strongly on time scales of the order of a hundredth of a second, and therefore has its strongest Fourier components around the 100 Hz region where the broadband detectors perform best. The primary waves and the growth of the memory are both cut off when the binary's neutron star or stars are tidally disrupted. The larger the radius of the neutron star, the sooner this occurs. As a result, the strength of the memory, and therefore the strength of the optimally filtered signal in the detectors, is quite sensitive to the neutron-star radius. Specifically, the memory's strength is of the order of (distance to Earth)$^{-1}\times$ (the energy carried off by the primary wave burst). The energy carried off is of the order of the binary's gravitational binding energy at tidal disruption, that is $\sim \mu M/2R$, where $\mu$ and $M$ are the binary's reduced and total masses and $R$ is the neutron-star radius. Therefore, the memory behaves like $h \propto 1/R$. Unfortunately, as we will see in Sec. IV, for the case of binaries containing a neutron star, the memory is likely to be too weak for either detection or measurement, even when fairly optimistic assumptions are made concerning event rate and detector sensitivities. For some two-black-hole binaries, however, the memory might be just detectable.

The paper is organized as follows. In Sec. II, I sketch a derivation, based on the quadrupole moment formalism, of the time evolution of the wave's memory. The final formula for the memory, Eq. (5), is somewhat inaccurate, because of post-Newtonian and higher order relativistic effects. The magnitude and sign of the errors are discussed at the end of Sec. IV, and are seen not to change any of the paper's conclusions. In Sec. III, I discuss the method of optimal signal processing that would be used, with a broadband LIGO and/or VIRGO detector, to search for the memory and measure its size. I also write down the formulas for the memory's signal-to-noise ratio and for the detector noise spectrum (assuming an "advanced" LIGO detector [1]) which goes into the signal-to-noise formula. In Sec. IV, I describe two different calculations of the signal-to-noise ratio which I have carried out, one based largely on numerical integrations, the other on analytical approximations. I give analytical formulas and a graph from which one can infer the $S/N$ for any desired binary. In Sec. V, I apply my results to specific examples of NS-NS, NS-BH, and BH-BH binaries. In Sec. VI, I discuss the implications of my results. Throughout the paper I use units in which Newton's constant of gravitation and the speed of light are unity, i.e., $G = c = 1$.

## II. THE FORM OF THE CHRISTODOULOU MEMORY

Thorne gives an expression for the net Christodoulou memory, when it has ceased growing, in terms of the total energy per unit solid angle, $dE/d\Omega'$, carried off by the primary waves [5]. Since the memory, at any moment of retarded time during its growth, is produced by the stress energy of all the waves emitted up until then, one can obtain an expression for the time-evolving memory $h(t)$ by replacing $dE/d\Omega'$ in Thorne's formula with $\int_{-\infty}^{t}(d^2E/d\Omega' dt')dt'$. The result is

$$h(t) = \frac{2}{r}\int_{-\infty}^{t}\int \frac{d^2E}{dt'd\Omega'}(1+\cos\theta')e^{2i\phi'}d\Omega' dt'. \tag{1}$$

The mathematical and geometric conventions are as follows: $h(t)$ is a complex gravitational wave field at Earth, equal to $h_+ + ih_\times$, with the transverse $+$ axes chosen arbitrarily and the $\times$ axes at 45° to it; the direction from source to Earth is the $z'$ axis (the polar axis) and the $+$ axes are taken to be the $x'$ and $y'$ axes; $\theta'$ and $\phi'$ are the polar coordinates that correspond to this source-based Cartesian system ($x' = r\sin\theta'\cos\phi'$, $y' = r\sin\theta'\sin\phi'$, $z' = r\cos\theta'$); the angular integral is over the direction, in terms of $\theta'$ and $\phi'$, of emission of the primary waves, and $r$ is the distance from source to Earth. The power radiated by the binary into unit solid angle has been evaluated by many researchers, for instance, Peters and Mathews [12], using the quadrupole moment formalism. It is, after averaging over one complete orbit,

$$\frac{d^2E}{d\Omega' dt} = \frac{1}{2\pi}\frac{\mu^2M^3}{a^5}(1+6\cos^2\theta+\cos^4\theta), \tag{2}$$

where $a$, the orbital radius, shrinks due to radiation reaction in a manner given by

$$a = \left(\frac{256}{5}\mu M^2 t\right)^{1/4} \tag{3}$$

where $t$ is the time until final coalescence, assuming the system consists of two idealized point masses. In these equations, $\mu = m_1m_2/(m_1+m_2)$ is the reduced mass, $M = m_1 + m_2$ is the total mass of the system, and $\theta$ (not to be confused with $\theta'$) is the angle that the primary direction of emission ($\theta'$,$\phi'$) makes with the binary's rotation axis.

To simplify the evaluation of the angular integral in Eq. (1), orient the $x'$,$y'$ axes so that the binary's rotation axis lie in the $x'$-$z'$ plane, and denote by $\iota$ the angle between the rotation axis and the direction to earth (the $z'$ direction). Then

$$\cos\theta = \sin\iota\sin\theta'\cos\phi' + \cos\iota\cos\theta'. \tag{4}$$

Inserting Eqs. (2)–(4) into Eq. (1) and evaluating the angular integral, one obtains the following expression for the memory:

$$h(t) = \frac{3}{32}\left(\frac{5\mu^3M^2}{r^4t}\right)^{1/4}\left(1-\frac{\sin^2\iota}{18}\right)\sin^2\iota. \tag{5}$$

This expression agrees, except for a factor of 2, with the similar equation derived by Wiseman and Will [6].

Note that $h(t) \equiv h_+ + ih_\times$ is real, so the simplifying choice of orientation for the observer's $x'$,$y'$ axes has made $h_\times$ vanish and left $h(t) = h_+(t)$. Note also that the memory increases as the time to coalescence $t$ decreases. The objects involved are not point masses, however. At some point, say, at time $t_k$, tidal forces disrupt them, they begin to merge, and the abrupt reduction of energy emitted in the burst, which is likely to occur within

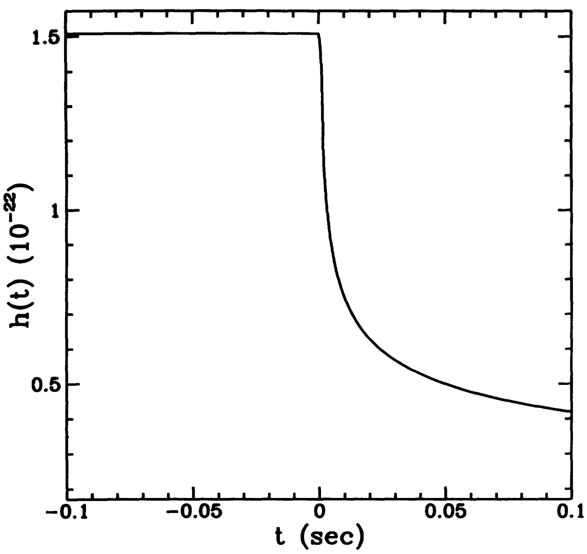


FIG. 1. A graph of the data train representing the memory. The dependence of $h(t)$ is $t^{-1/4}$ for $t \geq t_k$ and it is constant after this. (Recall that $t$ is the time *until* final coalescence in the ideal, Newtonian, point mass limit.) The discontinuity at $t_k$ is rounded off by an elliptical function in the numerical calculation, in order to model the actual turn-off of the wave, which takes place over at least an orbital period of the motion. In this figure, the round-off occurs over 1/500th of a second. Both this and Fig. 2 illustrate the case of two $10M_\odot$ black holes at a distance of 200 Mpc, with $t_k = 0.0006$ sec [the case in Eq. (28)].

about one orbital period [13–15], causes the memory to stop growing. Thereafter, it retains the amplitude it had at $t = t_k$.

The waveform in Eq. (5), with its growth terminated at $t = t_k$, is plotted in Fig. 1. Because $h_\times = 0$, the signal sensed by the detector is $h_d = F_+ h_+$, where $F_+ \leq 1$ is the detector's quadropolar antenna beam pattern function, which depends on the orientation of the detector to the incoming wave [Eq. (104a) and Fig. 9.9 of Ref. [16]]. If the detector happens to be so oriented as to maximize $h_d$ then $F_+ = 1$ and therefore $h_d = h(t)$.

For comparison with the time evolution $h(t)$ of the memory, Eq. (5), it will be important to know the time evolution of the frequency $f_p$ of the primary waves, which is twice the orbital frequency [16]. Therefore

$$f_p = \frac{1}{\pi M}\left(\frac{5M}{256\mu}\right)^{3/8}\left(\frac{M}{t}\right)^{3/8}$$
$$= 3.387 \times 10^3 \text{ Hz}\left(\frac{M}{4\mu}\right)^{3/8}\left(\frac{M_\odot}{M}\right)^{5/8}\left(\frac{1 \text{ msec}}{t}\right)^{3/8}. \quad (6)$$

Recall that for a binary, $M/4\mu \geq 1$.

## III. ANALYZING THE SIGNAL IN THE DETECTOR

In order to estimate the ability of a detector, such as LIGO, to detect the memory, it is necessary to calculate the signal-to-noise ratio of the signal in the detector. An experimenter, knowing the detector's noise spectrum, constructs a filter which is designed to let through the signal, while blocking out as much of the noise as possible. The Wiener optimal filter for a signal $h(t)$, seen in a detector with a one-sided noise spectrum $S_h(f)$, is a function $k(t)$, whose Fourier transform is related to the Fourier transform of the signal by

$$\tilde{k}(f) = \frac{\tilde{h}(f)}{S_h(f)}. \quad (7)$$

The filter is therefore a function similar to the signal, except that those frequencies which are noisy in the detector are suppressed. This is illustrated by the numerically derived graph of the filter function $k(t)$ in Fig. 2. The signal-to-noise ratio after optimal filtering, and taking account of the detector's beam pattern, is given by [17]

$$\left(\frac{S}{N}\right)^2 = 2F_+^2 \int_{-\infty}^{\infty} k(t)h(t)dt \quad (8a)$$

or

$$\left(\frac{S}{N}\right)^2 = 4F_+^2 \int_0^{\infty} \frac{|\tilde{h}(f)|^2}{S_h(f)}df. \quad (8b)$$

In evaluating $S/N$ for the Christodoulou memory, I have used the noise spectrum projected for the "advanced" detectors in LIGO, which may be in operation by the middle of the next decade. I have approximated their noise spectrum in the following way.

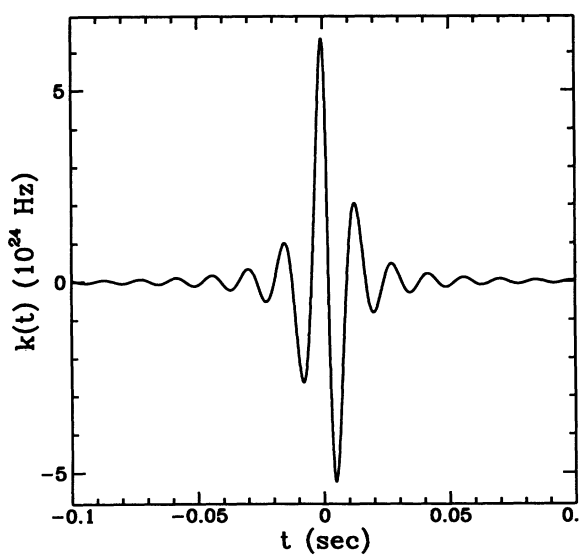


FIG. 2. A graph of the filter for the signal in Fig. 1, derived numerically, using a fast Fourier transform. Notice that this function, the integral of whose product with $h(t)$ gives the signal-to-noise ratio, is appreciably nonzero only in the 0.01 sec or so around $t_k$. Thus the signal-to-noise ratio derived from this filter depends almost entirely on the part of the signal where the rate of change is about 100 Hz, near the peak of the LIGO sensitivity.

$$S_h(f) = \begin{cases} (h_m^2/f_m)(f/f_m)^2, & f \geq f_m, \\ (h_m^2/f_m)(f/f_m)^{-4}, & f < f_m, \end{cases} \tag{9}$$

where $h_m = 1.0 \times 10^{-23}$ and $f_m = 70$ Hz [1].

This approximation to $S_h(f)$ ignores a seismic cutoff (in which $S_h$ rises very rapidly below 10 Hz) because it turns out to have a negligible influence on the signal-to-noise ratio. (Without the cutoff, $S_h \propto f^{-4}$ and $|\tilde{h}|^2 \propto f^{-2}$ at small $f$ [Eq. (14) below], so the cutoff produces a correction $\sim$ (10 Hz/70 Hz)$^3$, which is much less than one percent.) Between the seismic cutoff and 70 Hz (the optimal frequency), the noise is principally due to thermal noise in the test mass suspension, and above 70 Hz to photon shot noise in the interferometer beam [1]. The frequency of the primary gravitational waves from the coalescing binary, which is approaching the kilohertz range in the last 1/10 second of the inspiral, will fall outside of the LIGO detection window at that time. Although the Christodoulou memory is a dc signal, and therefore unobservable because of the low-frequency noise, its growth is detectable by observing it over the last $\sim$ 1/10 sec of the burst. At the very end of the signal the rate of change is quick enough, so that a great part of the signal can be seen on the time scale of LIGO's optimal frequency, which is an important advantage for detection, as pointed out by Braginsky and Thorne [3]. To the observers, the primary wave burst is a precursor to the memory it generates, because the two are detectable by them at different times.

Another type of detector which may go into operation in the next century is a space-based interferometer, such as the recently proposed Laser Interferometer Space Antenna (LISA) [18]. In the absence of seismic noise, such detectors would be much more sensitive to low frequency waves, and could therefore see more of the growth of the memory than any earth-based detector. Its noise spectrum can be modeled in the following way [18]:

$$S_h(f) = \begin{cases} (h_m^2/f_1)(f/f_2)^2, & f \geq f_2, \\ (h_m^2/f_1), & f_1 \leq f < f_2, \\ (h_m^2/f_1)(f/f_1)^{-4}, & f < f_1, \end{cases} \tag{10}$$

where $h_m = 3 \times 10^{-23}$, $f_1 = 10^{-3}$ Hz, and $f_2 = 10^{-1}$ Hz. Up to $f_1$ the noise is due to residual effects which perturb the spacecraft's inertial motion. Above $f_1$ it is due to photon shot noise, and the disimprovement beyond $f_2$ occurs because of the waves becoming shorter than the interferometer arm.

The binary coalescences which the LIGO and/or VIRGO detectors will be searching for are thought to be very rare: a few per year to a few per day at the strengths detectable by these instruments. In order to be certain, with 99% confidence, that an observed primary-wave signal is not due to the detector's internal Gaussian noise, one must require that a signal-to-noise ratio of 6 or better be registered in two independent detectors simultaneously [8]. However, once the primary waves have been discovered, one can predict from them, to within an accuracy of about 0.001 sec or less, the time at which the memory will register most strongly in the detector. With this knowledge, there are no free parameters to be solved for in the memory measurement, and the detector's optimally filtered output will therefore be a single number. The noise component of this number should be Gaussianly distributed, so the usual Gaussian criteria for detection apply. For measurement by a single detector, an observed signal with $S/N = 1$ is real with 68% confidence, which is increased to 95% confidence if $S/N = 2$ and to $99\frac{3}{4}$% confidence if $S/N = 3$. More to the point, if both LIGO detectors register $S/N = 2$ then one can be $99\frac{3}{4}$% confident that the memory was detected. Therefore, in the following section, I shall regard $S/N = 2$ as a reasonable criterion for detectability by LIGO. In the case of a space-based detector, where only one instrument would be in operation, $S/N \geq 3$ would be required, without a coincident detection by another system.

## IV. THE SIGNAL TO NOISE RATIOS

I have computed the signal-to-noise ratio $S/N$ for the Christodoulou memory produced by NS-BH, NS-NS, and BH-BH binaries, under a variety of assumptions about the time at which tidal disruption or coalescence terminates the primary waves. In order to compute the $S/N$, one needs to find the Fourier transform of the signal $\tilde{h}(f)$. One method I have used is to calculate this function and the signal to noise itself numerically, using a fast Fourier transform algorithm taken from Ref. [19]. In an idealized case, with the cutoff time $t_k \leq 0.1$ msec, so that $2\pi f_m t_k$ (the detector's optimal angular frequency times the cutoff time) can be regarded as a small parameter, I have been able to do the whole calculation analytically, in the following way.

Write the signal in the form

$$h(t) = \begin{cases} h_k(t_k/t)^{1/4}, & t \geq t_k \\ h_k, & t \leq t_k \end{cases}, \tag{11}$$

where $h_k = h(t_k)$ is given by Eq. (5):

$$h_k = \frac{3}{32}\Big(\frac{5\mu^3 M^2}{r^4 t_k}\Big)^{1/4}\Big(1 - \frac{\sin^2\iota}{18}\Big)\sin^2\iota$$
$$= 2.56 \times 10^{-24}\Big(\frac{4\mu}{M}\Big)^{3/4}\Big(\frac{M}{M_\odot}\Big)^{5/4}\frac{1}{x^{1/4}}$$
$$\times\Big(\frac{200\ \text{Mpc}}{r}\Big)\Big(1 - \frac{\sin^2\iota}{18}\Big)\sin^2\iota. \tag{12}$$

Here $x = 2\pi f_m t_k$, so that $t_k$ is rescaled by LIGO's optimal frequency, $f_m$. The distance $r$ is measured in units of 200 Mpc for reasons that will become evident below. Then one can reexpress Eq. (11) as

$$h(t) = h_k(t_k/t)^{1/4}\Theta(t - t_k) + h_k\Theta(t_k - t), \tag{13}$$

where $\Theta$ is the step function, $\Theta(\xi) = 1$ for $\xi > 0$ and 0 for $\xi \leq 0$. The Fourier transforms of the step function and the power law $1/t^{1/4}$ are well known (see, for instance, the Bateman papers [20]). Using them and the convolution theorem it is possible to compute the Fourier transform of Eq. (13). When the convolution is evaluated as a power series in $2\pi f t_k$, the result is

$$\tilde{h}(f) = -\frac{ih_k}{2\pi f}\Big(1 + i\alpha(2\pi f t_k)^{1/4} + i(1 - \tfrac{4}{3}\gamma)2\pi f t_k - \tfrac{1}{2}(1 - \tfrac{8}{7}\gamma)(2\pi f t_k)^2 + \dots\Big) \tag{14}$$

where $\alpha = \rho + i\sigma$ with $\rho = 0.469$ and $\sigma = 1.13$, and where $\gamma = 6.28$, and terms which are higher order in $f t_k$ are neglected. See the appendix where the derivation of this equation is explained in more detail, and the exact values of $\alpha$ and $\gamma$ are given.

Using Eqs. (8b) and (14) and the noise spectrum defined in Eq. (9), it is straightforward to calculate the signal-to-noise ratio for the memory, seen in an advanced LIGO detector. The result is

$$\left(\frac{S}{N}\right)^2 \equiv \frac{2}{3}F_+^2\left(\frac{h_k}{\pi h_m}\right)^2\Sigma^2 = \frac{2}{3}F_+^2\left(\frac{h_k}{\pi h_m}\right)^2\left[1 - \frac{288}{143}\sigma x^{1/4} + \frac{36}{35}|\alpha|^2 x^{1/2} + \frac{288}{119}\rho\left(1 - \frac{4}{3}\gamma\right)x^{5/4} - \frac{96}{35}\gamma\left(1 - \frac{7}{6}\gamma\right)x^2 + \cdots\right]. \tag{15}$$

Again, higher order terms are neglected. The coefficients in this power series in $x = 2\pi f_m t_k$ are mostly of order unity, except for the $x^2$ term. Therefore, for an error of less than 10% to order $x^2$, $x = 2\pi f_m t_k$ must be $\leq 0.04$, which means (since $f_m = 70$ Hz), that $t_k$ must be $\leq 10^{-4}$ sec. The next large coefficient term is the $x^4$ term, but at that order high frequency contributions from the discontinuous derivative at $t = t_k$ introduce infinities into the series. So one cannot improve the range of validity of the analytic approximation by simply going to higher order in the expansion, without drastically altering and complicating the calculation. In the numerical case, however, it is straightforward to round off the sharp edge at the cutoff (which is unphysical), and so the numerical results are preferred for $t_k \geq 0.1$ msec. For a comparison of the numerical and analytical results, and a graph of $\Sigma$, as defined in Eq. (15), see Fig. 3.

Since the detector's minimum noise level is $h_m = 10^{-23}$, and since the memory's full strength $h_k$ is given by Eq. (12), in the limit $2\pi f_m t_k \ll 1$ the signal-to-noise ratio (15) becomes

$$\begin{aligned}\frac{S}{N} &= \left(\frac{2}{3}\right)^{1/2}\left(\frac{h_k}{\pi h_m}\right)F_+ \\ &= 0.0665F_+\left(\frac{4\mu}{M}\right)^{3/4}\left(\frac{M}{M_\odot}\right)^{5/4}\left(\frac{200\ \mathrm{Mpc}/r}{(2\pi f_m t_k)^{1/4}}\right)\left(1 - \frac{\sin^2\iota}{18}\right)\sin^2\iota.\end{aligned} \tag{16}$$

This $S/N$ increases monotonically with decreasing cutoff time $t_k$, as one would expect. The same calculation can be made in the case of the space-based detector, using the noise spectrum in Eq. (10). The result is

$$\left(\frac{S}{N}\right)^2 = \frac{4}{3}\left(\frac{F_+h_k}{\pi h_m}\right)^2\left[1 - \frac{32}{13}\sigma x_1^{1/4} + \frac{12}{7}|\alpha|^2 x_1^{1/2} - \frac{96}{17}\rho\left(1 - \frac{4}{3}\gamma\right)x_1^{5/4} + \frac{96}{105}\gamma\left(1 - \frac{7}{6}\gamma\right)x_1^2 + \dots - \frac{f_1}{f_2}\left(\frac{1}{2} - \frac{16}{11}\sigma x_2^{1/4} + \frac{6}{5}|\alpha|^2 x_2^{1/2} - \frac{48}{7}\rho\left(1 + \frac{4}{3}\gamma\right)x_2^{5/4} - \frac{16}{7}\gamma\left(1 - \frac{7}{6}\gamma\right)x_2^2 + \dots\right)\right]. \tag{17}$$

Here, the expansion is in terms of $x_1 = 2\pi f_1 t_k$ and $x_2 = 2\pi f_2 t_k$.

The cutoff time, $t_k$, is constrained in three different ways. One is by the total amount of energy radiated in the primary waves, which I shall refer to as $E_{\mathrm{GW}}$. The relation is

$$\left(\frac{t_k}{M}\right)^{1/4} = \frac{1}{32}\frac{\mu}{E_{\mathrm{GW}}}\left(\frac{5M}{\mu}\right)^{1/4}. \tag{18}$$

Since $E_{\mathrm{GW}}$ cannot exceed $\mu$ and will typically be less, and since $\mu$ cannot exceed $M/4$, then

$$\begin{aligned}t_k &= 9.4 \times 10^{-11}\mathrm{sec}\left(\frac{M}{M_\odot}\right)\left(\frac{\mu}{E_{\mathrm{GW}}}\right)^4\frac{M}{4\mu} \\ &\geq 9.4 \times 10^{-11}\mathrm{sec}\left(\frac{M}{M_\odot}\right).\end{aligned} \tag{19}$$

A second constraint on $t_k$ comes from the total number of orbits left until $t = 0$, if there were no cutoff, which is

$$N_{\mathrm{orb}} = \frac{4}{5}f_p t_k = \frac{4}{5\pi}\left(\frac{5M}{256\mu}\right)^{3/8}\left(\frac{t_k}{M}\right)^{5/8}. \tag{20}$$

This number must exceed unity if the analysis is to make any sense, since the quadrupole formalism requires averaging over an orbit. This constraint says that

$$\begin{aligned}t_k &= \left(\frac{5\pi}{4}\right)^{8/5}\left(\frac{64}{5}\right)^{3/5}\left(\frac{4\mu}{M}\right)^{3/5}N_{\mathrm{orb}}^{8/5}M \\ &> 2 \times 10^{-4}\mathrm{sec}\left(\frac{4\mu}{M}\right)^{3/5}\left(\frac{M}{M_\odot}\right).\end{aligned} \tag{21}$$

This constraint (21) is more severe than (19) for binaries with realistic mass ratios, $4\mu/M > 2.77 \times 10^{-11}$ (i.e., $m_2/m_1 > 6.93 \times 10^{-12}$), while (19) is more severe in the unrealistic regime of very extreme mass ratios, where $4\mu/M > 2.77 \times 10^{-11}$ (i.e., $m_2/m_1 < 6.93 \times 10^{-12}$). Therefore, in cases of interest to us, (21) is always the active constraint.

The third restriction is the actual relation between $t_k$ and the size of the binary's components, which depends on the onset of tidal disruption. If disruption begins as the two bodies first touch, then

$$t_k = \frac{5}{64}\left(\frac{M}{4\mu}\right)\left(\frac{r_1+r_2}{M}\right)^4 M$$
$$= 3.85 \times 10^{-7}\text{sec}\left(\frac{M}{4\mu}\right)\left(\frac{r_1+r_2}{M}\right)^4\left(\frac{M}{M_\odot}\right), \qquad (22)$$

where $r_1$ and $r_2$ are the radii of the two bodies [12]. In the most extreme conceivable case, where $r_1 + r_2 = M$ (so the sum of the bodies' physical radii is equal to the sum of one-half their Schwarzschild radii, recall that a rapidly spinning hole has a radius equal to half of the Schwarzschild radius), this gives $t_k \geq 3.85 \times 10^{-7}(M/4\mu)(M/M_\odot)$.

In the LIGO and/or VIRGO frequency band one deals with binaries for which $4\mu \geq 1M_\odot$, $M \leq 300M_\odot$, so (21) is generally the active constraint on $t_k$. For LISA the mass ratio can be much more extreme, so either (21) or (22) can be the relevant constraint.

## V. EXAMPLES AND APPLICATIONS

Narayan, Piran, and Shemi [21] and Phinney [22] have estimated a coalescence rate of a few per year for neutron-star–black-hole binaries at a distance of 200 Mpc from Earth. To model this type of source I use $m_1 = 10M_\odot$, $m_2 = 1.4M_\odot$. One can derive an estimate of the total energy radiated in the primary wave from the energy of the last stable circular orbit of the neutron star (treated as a point mass) about the black hole, which is assumed to be maximally rotating (in order to maximise the $S/N$). This gives $E_{\rm GW} = 0.4828\mu$ [23], which implies [from Eqs. (18) and (15)] that the signal-to-noise ratio would be

$$\frac{S}{N} = 2.0\left(\frac{200\ \text{Mpc}}{r}\right)F_+\left(1 - \frac{\sin^2\iota}{18}\right)\sin^2\iota. \qquad (23)$$

The cutoff time in this case is unrealistically small ($t_k \approx 10\ \mu$sec, which is much less than an orbital period), because of the inadequecy of the quadrupole approximation. Using my numerical calculation of $S/N$ (see Fig. 3) and still taking a deliberately exaggerated case, if the two bodies begin to merge as they touch and the black hole is nonrotating, then the memory should stop growing at $a \approx 40$ km (in fact, the last stable circular orbit in this case is at $a \approx 90$ km). At this stage the frequency of the primary waves would have reached about 1500 Hz, and $t_k = 0.00036$ sec. In this extreme case the signal-to-noise ratio in an advanced LIGO detector would be

$$\frac{S}{N} = 0.45\left(\frac{200\ \text{Mpc}}{r}\right)F_+\left(1 - \frac{\sin^2\iota}{18}\right)\sin^2\iota. \qquad (24)$$

If the black hole is nearly maximally rotating, as seems likely for one which is in a binary system, where it would be spun up by infalling debris, then its horizon is at one-half the Schwarzschild radius. Still assuming a radius of 10 km for the neutron star, merger could commence no later than when they are touching at $a \approx 25$ km. The signal-to-noise ratio then amounts to

$$\frac{S}{N} = 1.1\left(\frac{200\ \text{Mpc}}{r}\right)F_+\left(1 - \frac{\sin^2\iota}{18}\right)\sin^2\iota. \qquad (25)$$

Since $F_+ \leq 1$, and since $r \approx 200$ Mpc is actually a rough lower bound for how far one must look to see several NS-BH coalescences per year [21,22], and since all of these very optimistic examples give $S/N$ of near or less than two, the prospects are rather poor for advanced LIGO and/or VIRGO detectors to see the Christodoulou memory from NS-BH binaries.

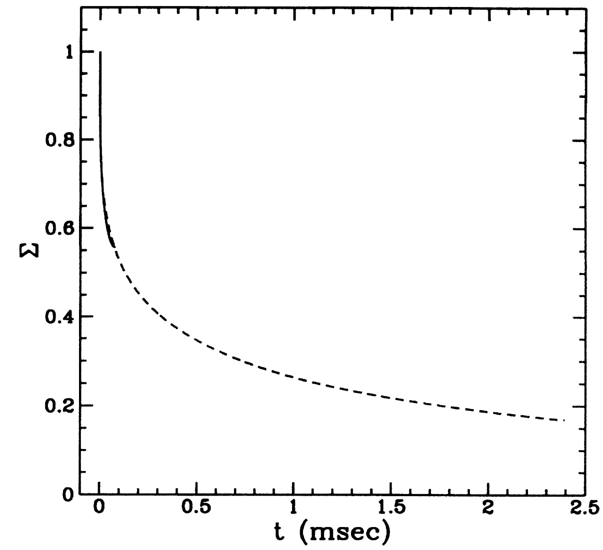


FIG. 3. A graph of $\Sigma$ [defined in Eq. (15)], for an advanced LIGO detector with the noise spectrum given in Eq. (9), versus cutoff time $t_k$. Note that it is a monotonically decreasing function of the cutoff time. The solid line in the graph was derived analytically, from the series in Eq. (15), and the dotted line represents data which were calculated numerically, by means of a fast Fourier transform, and then integrating the product of the signal and filter functions (such as those shown in Figs. 1 and 2) via Eq. (8a).

If a sizeable fraction of the main-sequence progenitors of NS-NS binaries actually make such binaries when they die, rather than disrupting during a supernova outburst or dying via some other route, then the NS-NS coalescence rate could be several per year at distances as close as $\sim 30$ Mpc [22,24]. With $m_1 = m_2 = 1.4M_\odot$, and following Rasio and Shapiro [13] in supposing a dramatic reduction in power radiated at $a \approx 20$ km, then

$$\frac{S}{N} = 0.75\left(\frac{30\ \text{Mpc}}{r}\right)F_+\left(1 - \frac{\sin^2\iota}{18}\right)\sin^2\iota. \qquad (26)$$

Thus, even under these most optimistic assumptions, the prospects for seeing NS-NS memories are dim.

In the case of two $10M_\odot$ black holes, one can set an upper bound on the energy radiated in the primary wave by demanding that the total surface area of the holes be conserved. For two nonrotating holes this sets $E_{\rm GW} = 0.293M$, where $M$ is the total mass. The memory generated by this much energy in the primary wave would produce a signal to noise in the detector (derived analytically) of

$$\frac{S}{N} = 14.0\left(\frac{200\text{ Mpc}}{r}\right)F_+\left(1 - \frac{\sin^2 \iota}{18}\right)\sin^2 \iota. \tag{27}$$

The cutoff time in this case, $t_k < 1\mu$s, is even more unrealistic then in the first cited NS-BH case above. In an effort to estimate $S/N$ with a more realistic coalescence time take $t_k = 0.0006$ sec, for which the (numerical) result is

$$\frac{S}{N} = 1.3\left(\frac{200\text{ Mpc}}{r}\right)F_+\left(1 - \frac{\sin^2 \iota}{18}\right)\sin^2 \iota. \tag{28}$$

Since 200 Mpc is a lower bound on how far one must look to see several BH-BH coalescences per year [22,24], the prospects are modestly hopeful (but only modestly) that LIGO and/or VIRGO might one day detect the memory of a BH-BH coalescence.

As was discussed earlier, space-based interferometers are more sensitive to very low frequency waves, and might be expected, for that reason, to be better able to detect the Christodoulou memory than ground-based detectors. However, they have no particular advantage in the case of neutron-star binaries, where the bulk of the growth in the memory takes place over a timescale favorable to LIGO. In Eq. (17), the dominant terms in the series expansion depend on $x_1 = 2\pi f_1 t_k$, where $f_1 = 10^{-3}$ Hz. If $t_k < 0.16$ sec, so $x_1 \leq 10^{-3}$, then all terms except the first three are negligible and can be ignored. Assuming the constraint in Eq. (21) applies to the cutoff time, and letting $N_{\rm orb} = 1$, so that merger is presumed to take place at the latest possible time (within the framework of the approximation scheme), then $t_k = (5\pi/4)^{8/5}(256/5)^{3/5}M_c$ and so

$$t_k = 4.66 \times 10^{-4}\frac{M_c}{M_\odot}, \tag{29}$$

where $M_c = \mu^{3/5}M^{2/5}$ is the chirp mass of the binary. In this way, a formula can be derived relating the signal-to-noise ratio generated by a binary with chirp mass $M_c$ at a distance $r$ from the detector, to these two quantities, assuming that merger takes place when $N_{\rm orb} = 1$. The approximation should hold good up to $M_c/M_\odot \approx 300$:

$$\frac{S}{N} \approx 0.1\frac{M_c/M_\odot}{r/200\text{ Mpc}}F_+\left(1 - \frac{\sin^2 \iota}{18}\right)\sin^2 \iota\left[1 - 0.115\left(\frac{M_c}{M_\odot}\right)^{1/4} + 0.0044\left(\frac{M_c}{M_\odot}\right)^{1/2}\right]^{1/2} \tag{30}$$

In the case of a $10M_\odot$ black hole and a neutron star at $r = 200$ Mpc, then $M_c = 3M_\odot$ and so $S/N \leq 0.3$. A NS-NS binary has a chirp mass of $M_c = 1.219M_\odot$, and at $r = 30$ Mpc its signal-to-noise ratio would be $S/N \leq 0.75$. Two $10M_\odot$ black holes have a chirp mass of $M_c = 8.7M_\odot$. At a range of 200 Mpc, their signal-to-noise ratio would be $S/N \leq 0.8$. Finally, a binary with chirp mass of $100M_\odot$ at a distance of 1 Gpc would produce a signal-to-noise ratio of $S/N \leq 1.6$. This last example begins to approach the sort of large mass systems which are of special interest to space-based detectors such as LISA, but these systems are of no interest from the point of view of measuring neutron-star radii. Obviously LISA would be no more use than LIGO for estimating neutron-star radii, but it might well be capable of detecting the memory produced by very massive binaries, if, for instance, mergers between super-massive black holes are sufficiently common within a few Gpc.

In this paper I have made use of a simplistic quadrupole-moment (i.e., slow-motion) calculation of the memory. Would a more relativistic approach increase the memory, thereby increasing the odds of detection? Finn [25] has made a detailed numerical calculation of the power emitted by systems consisting of low mass bodies in equatorial orbits around massive rotating black holes. As it happens, his figures indicate that the quadrupole approximation consistently overestimates the power emitted in the burst by a modest factor, and therefore the memory would, in reality, be modestly weaker than I have made it. More specifically, in the case of a small body in a prograde orbit around a nearly maximally rotating Kerr black hole, Finn finds that for an orbital radius of $10M$, where $M$ is the mass of the hole, the loss of energy due to gravitational-wave emission is roughly 90% of its value as derived by the quadrupole formula. When the particle reaches the last stable circular orbit at $r = M$, his result has fallen to about 50% of the quadrupole value. The nearly Newtonian approximation can therefore be taken as at least a rough guide to the results of a more realistic calculation.

## VI. CONCLUSIONS

In all of the cases considered above involving neutron stars, the signal-to-noise ratio fell on or below the threshold for detection ($S/N = 2$). In addition, one has to remember that the indication is that the true memory would be somewhat smaller, because of the quadrupole formula's overestimation of the energy emitted by the binary in the last stages of coalescence. In fact, since detection depends largely on the rate of growth of the memory in the last split second of the inspiral, it seems likely that the signal-to-noise ratio, in practice, could be as low as close to one-half this paper's estimated value, in the case of a rapidly spinning black hole. Therefore the chances of even an advanced LIGO interferometer detecting the Christodoulou memory, except serendipitously, from sources such as these appears to be small. Certainly, there is little chance of using the signal to estimate neutron-star radii, unless coalescence rates have been drastically underestimated.

Fortunately another, more promising, method of measuring neutron-star radii has been proposed [8] . This scheme involves the use of several narrow band detectors with optimal frequencies staggered around 1 kHz, which would register the strength of the primary signal (if any) as it passes through their frequency. The waves' cutoff frequency can then be estimated from their responses, and therefore the neutron-star radii can be deduced [8]. A companion paper [11] gives a quantitative description and evaluation of this method.

The results given above indicate (in agreement with a cruder estimate by Thorne [5]), that the memory from coalescing binaries consisting of two large black holes within $\approx 200$ Mpc of earth might be detectable by LIGO. But the lack of any detailed understanding of the behavior of BH-BH binaries makes any prediction uncertain. My results merely demonstrate that the memory from such systems *may* be strong enough to be seen by very sensitive detectors. It seems likely that at least the existence of the memory might be confirmed by a particularly strong event.

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## APPENDIX

Let $H(t) = h'(t)\Theta(t - t_k)$, where $h'(t) = c/t^{1/4}$ ($c = h_k t_k^{1/4}$) for $t > 0$ and $= 0$ for $t < 0$. This is the first term in Eq. (13), the Fourier transform of which is needed to derive Eq. (14). The Fourier transform of $H(t)$ is given by

$$\tilde{H}(\omega) = \int_{-\infty}^{\infty} \tilde{h}'(\omega')\tilde{\Theta}(\omega - \omega')d\omega' \tag{A1}$$

(where $\omega = 2\pi f$), by the convolution theorem. From the Bateman papers [20], $\tilde{h}'(\omega) = \alpha/\omega^{3/4}$ for $\omega > 0$ and $= \alpha^*/|\omega|^{3/4}$ for $\omega < 0$, where

$$\alpha = \frac{\pi}{2}\frac{1}{\Gamma(\frac{1}{4})}\sec\frac{\pi}{8} + i\Gamma(\tfrac{3}{4})\cos\frac{\pi}{8}. \tag{A2}$$

Therefore

$$\begin{aligned}\tilde{H}(\omega) = {} & ic\alpha\int_0^{\infty}\frac{e^{i(\omega-\omega')t_k}}{\omega-\omega'}\frac{d\omega'}{\omega'^{3/4}} \\ & + ic\alpha^*\int_{-\infty}^{0}\frac{e^{i(\omega-\omega')t_k}}{\omega-\omega'}\frac{d\omega'}{|\omega'|^{3/4}}\end{aligned} \tag{A3}$$

which implies that

$$\tilde{G}(\omega) \equiv \frac{d}{dt_k}\tilde{H}(\omega) = -\frac{\gamma c}{t_k^{1/4}}e^{i\omega t_k} \tag{A4}$$

where $\beta = \int_0^{\infty} e^{-ix}dx/x^{3/4}$ and $\gamma = \alpha\beta + \alpha^*\beta^* \approx 6.28$. From this it follows that

$$\tilde{H}(\omega) = \int \tilde{G}(\omega)dt_k + \tilde{F}(\omega), \tag{A5}$$

where $\tilde{F}(\omega)$ is independant of $t_k$. Now,

$$\begin{aligned}\int \tilde{G}(\omega)dt_k &= -\frac{c\gamma}{\omega^{3/4}}\int\frac{e^{i\omega t_k}}{(\omega t_k)^{1/4}}d(\omega t_k) \\ &= -\frac{c\gamma}{\omega^{3/4}}\Big[\frac{4}{3}(\omega t_k)^{3/4} + \frac{4}{7}i(\omega t_k)^{7/4} \\ &\quad + \cdots\Big],\end{aligned} \tag{A6}$$

which is zero as $t_k \to 0$. Therefore, $\tilde{F}(\omega) = \tilde{H}(\omega, t_k = 0) = \tilde{h}'(\omega)$, since $H(t, t_k = 0)$ is simply $h'(t)$. From this, and Eqs. (A5) and (A6), and the above expression for $\tilde{h}'(\omega)$, together with the Fourier transform of the second term in Eq. (13), one derives Eq. (14).

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